JOURNAL OF COMBINATORIAL THEORY, Series B 44, 253–262 (1988)

A Compactness Theorem for Perfect Matchings in Matroids

P. KOMJATH,* E. C. MILNER,† AND N. POLAT‡

*Department of Computer Science, Eotvos University, Budapest, P.O. Box 323, 1445, Hungary,

†Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4, and

‡Département de Mathématiques, Université Claude-Bernard (Lyon 1), Villeurbanne, France

Communicated by the Editors

Received December 4, 1985

A family, $\mathscr{E}$, of finite sets is said to be finite matching extendable (f.m.e) if, for every finite matching $\mathscr{M} \subseteq \mathscr{E}$ and every vertex $x \in \bigcup\mathscr{E} \setminus \bigcup\mathscr{M}$, there is $E \in \mathscr{E}$ such that $x \in E$ and $E \cap \bigcup\mathscr{M} = \varnothing$. It is shown that, if a matroid is f.m.e., then it has a perfect matching. This generalizes a well-known theorem of Nash–Williams which states that the edges of a graph can be covered by edge-disjoint circuits if every finite set of edges can be so covered. © 1988 Academic Press, Inc.

1. INTRODUCTION

A *matching* in a family of sets $\mathscr{E}$ is a subfamily $\mathscr{M} \subseteq \mathscr{E}$ whose members are pairwise disjoint. We say that $\mathscr{E}$ is *matchable*, or has a *perfect matching*, if there is a matching $\mathscr{M} \subseteq \mathscr{E}$ such that $\bigcup\mathscr{M} = \bigcup\mathscr{E}$. The family $\mathscr{E}$ is said to be *finitely matchable* (f.m.) if for every finite set $F \subseteq \bigcup\mathscr{E}$, there is a matching $\mathscr{M} \subseteq \mathscr{E}$ which *covers* F (i.e., $F \subseteq \bigcup\mathscr{M}$).

A well-known theorem of Nash–Williams [3] asserts that the edges of a graph may be covered by edge-disjoint circuits provided every finite set of the edges may be so covered. Thus, in our terminology, if $\mathscr{E}$ is the set of circuits of a graph (the elements of a member $E \in \mathscr{E}$ are the edges comprising E), then $\mathscr{E}$ is matchable iff $\mathscr{E}$ is finitely matchable. Sabidussi [5] asked if this compactness result of Nash–Williams is true for arbitrary matroids. In other words, if $\mathscr{E}$ is the set of circuits (minimal dependent sets) in a matroid, does finite matchability imply matchability. Unfortunately this is not the case and we give a simple example (Example 3.2) to illustrate this. However, Polat [4] showed that the Nash–Williams result does extend to arbitrary binary matroids.

In this paper we consider another finitary condition which is stronger than finite matchability. We say that $\mathscr{E}$ is *finite matching extendable* (f.m.e.) if for every finite matching $\mathscr{M} \subseteq \mathscr{E}$ and every vertex $x \in \bigcup\mathscr{E} \setminus \bigcup\mathscr{M}$, there is

0095-8956/88 $3.00

$E \in \mathscr{E}$ such that $x \in E$ and E is disjoint from $U\mathscr{M}$; in other words, the matching $\mathscr{M}$ can be extended to a matching which covers x. The main result of this paper is the following compactness theorem.

THEOREM 1. *A matroid is matchable if it is f.m.e.*

It is easy to construct examples of matroids which are matchable but not f.m.e. (Example 4.2). However, for binary matroids the conditions f.m. and f.m.e. are equivalent (Lemma 3.1), so the above theorem is a generalization of the results of Nash–Williams [3] and Polat [4].

We actually prove a more general result than Theorem 1 (Theorem 4.1). Our proof, which is self contained, uses some of the basic ideas from [3] and [4], but the rather complicated and clever technique developed by Nash–Williams [3] (and successfully applied in [4]) fails to work here. Our method of proof instead employs very general compactness techniques of the kind first used by Shelah [6].

2. NOTATION AND PRELIMINARY LEMMAS

We use standard set-theoretical notation. In particular, if κ is a cardinal κ^+ is the successor cardinal of κ, and $\mathrm{cf}(\kappa)$ denotes the cofinality of κ. Also, if A is any set, then $[A]^\kappa$ (resp. $[A]^{<\kappa}$, $[A]^{\leqslant\kappa}$) denotes the set of all subsets of A of cardinality κ (resp. $<\kappa$, $\leqslant\kappa$).

Throughout $\mathscr{E}$ denotes a family of finite, non-empty sets, and we put $V = U\mathscr{E}$. We call the elements of V the *vertices* of $\mathscr{E}$. If $S \subseteq V$, then the *restriction* of $\mathscr{E}$ to S is $\mathscr{E} \restriction S = \mathscr{E} \cap \mathscr{P}(S)$; and we define $\mathscr{E} \backslash S = \mathscr{E} \restriction (V \backslash S)$. A set $S \subseteq V$ is *covered* by $\mathscr{E}' \subseteq \mathscr{E}$ if $S \subseteq U\mathscr{E}'$. We denote by $\mathscr{E}_{\min}$ the subset of all minimal (under $\subseteq$) members of $\mathscr{E}$.

A family of sets $\mathscr{F}$ is called a *Δ-system* if there is a set $K \subseteq U\mathscr{F}$ such that $F_1 \cap F_2 = K$ for every pair of distinct members of $\mathscr{F}$; we call K the *kernel* of the Δ-system.

Let $\mathscr{E}$ be a family of finite, non-empty sets, $V = U\mathscr{E}$. For any infinite cardinal κ we denote by $\mathscr{E}^{(\kappa)}$ the set of all $F \in [V]^{<\omega} \backslash \{\varnothing\}$ such that F is the kernel of some Δ-system $\mathscr{E}' \subseteq \mathscr{E}$ of cardinality $|\mathscr{E}'| \geqslant \kappa$. We also define $\mathscr{E}^{\langle\kappa\rangle} = (\mathscr{E} \cup \mathscr{E}^{(\kappa)})_{\min}$. Finally, we define an equivalence relation $R(\kappa)$ on V so that $(x, y) \in R(\kappa)$ if and only if either $x = y$ or there is a finite sequence $A_0, A_1, ..., A_n$ of elements of $\mathscr{E}^{\langle\kappa\rangle}$ such that $x \in A_0$, $y \in A_n$ and $A_i \cap A_{i+1} \neq \varnothing$ for $0 \leqslant i < n$. A subset $S \subseteq V$ is *closed* under $R(\kappa)$ if $S = U\{[x]_{R(\kappa)} : x \in S\}$, where $[x]_{R(\kappa)}$ is the equivalence class of $R(\kappa)$ containing x.

LEMMA 2.1 [4, 4.3]. *If $\kappa = cf(\kappa) > \omega$, then $|[x]_{R(\kappa)}| < \kappa$ for each $x \in V$.*

Proof. It is enough to observe that the valency of each vertex in the hypergraph $\mathscr{H} = (V, \mathscr{E}^{\langle\kappa\rangle})$ is less than κ, since this implies that each connected component of $\mathscr{H}$ has cardinality less than κ. Suppose for contradiction that $x \in V$ belongs to κ members of $\mathscr{E}^{\langle\kappa\rangle}$. Then there is $F \in [V]^{\langle\omega\rangle}$ such that $x \in F$ and F is the kernel of some Δ-system, $\mathscr{F}$, of size κ in $\mathscr{E}^{\langle\kappa\rangle}$. But then F is the kernel of a Δ-system $\mathscr{E}'$ of size κ in $\mathscr{E}$, and this contradicts the fact that the members of $\mathscr{F}$ are minimal members of $\mathscr{E} \cup \mathscr{E}^{(\kappa)}$. ∎

If κ is an infinite cardinal, we say that the set system $\mathscr{E}$ has property $Q(\kappa)$ if, for every $E \in \mathscr{E}$ and each $x \in E$, there is $F \in \mathscr{E}^{\langle\kappa\rangle}$ such that $x \in F \subseteq E$. Also, we set $P(\mathscr{E}, \kappa) = \{A \subseteq V: \mathscr{E}^{(\kappa^+)} \cap \mathscr{P}(A) \subseteq (\mathscr{E} \restriction A)^{(\kappa)}\}$.

LEMMA 2.2. *If $\mathscr{E}$ is f.m.e. and has property $Q(\kappa^+)$ and if $A \in P(\mathscr{E}, \kappa)$ is closed under $R(\kappa^+)$, then $\mathscr{E} \restriction A$ is f.m.e. and $A = U(\mathscr{E} \restriction A)$.*

Proof. Let $\mathscr{M}$ be a finite matching in $\mathscr{E} \restriction A$ and let $x \in A \backslash U\mathscr{M}$. Since $\mathscr{E}$ is f.m.e. there is $E \in \mathscr{E}$ such that $x \in E$ and E is disjoint from $U\mathscr{M}$. By assumption, there is $F \in \mathscr{E}^{\langle\kappa^+\rangle}$ such that $x \in F \subseteq E$. Also, since A is closed under $R(\kappa^+)$ we have that $F \subseteq A$. If $F \in \mathscr{E}$, we are done. If not, then $F \in \mathscr{E}^{(\kappa^+)}$ and therefore $F \in (\mathscr{E} \restriction A)^{(\kappa)}$. Since $U\mathscr{M}$ is finite, $F \cap U\mathscr{M} = \varnothing$ and F is the kernel of a Δ-system of size κ in $\mathscr{E} \restriction A$, it follows that there is $E_1 \in \mathscr{E} \restriction A$ such that $F \subseteq E_1$ and $E_1 \cap U\mathscr{M} = \varnothing$. ∎

The following lemma provides a key step in our proof of the main result.

LEMMA 2.3. *Let κ be an infinite cardinal and let $A, A_\alpha \in [V]^{\leqslant\kappa}$ $(\alpha < \kappa)$. Then there is a set $B \in [V]^{\leqslant\kappa}$ such that* (i) *B is closed under $R(\lambda^+)$ for every infinite cardinal $\lambda \leqslant \kappa$,* (ii) *$A \subseteq B$ and* (iii) *$B \backslash A_\alpha \in \mathscr{P}(\mathscr{E}, \mu)$ for $\alpha < \kappa$ and μ an infinite cardinal such that $|A_\alpha| \leqslant \mu \leqslant \kappa$.*

Proof. Put $B_0 = A$. Let $n < \omega$ and suppose that we have already defined $B_n \in [V]^{\leqslant\kappa}$. For $\alpha < \kappa$ and $|A_\alpha| \leqslant \mu \leqslant \kappa$ put $\mathscr{E}(\alpha, \mu) = \mathscr{E}^{(\mu^+)} \cap [B_n \backslash A_\alpha]^{<\omega}$. For each $F \in \mathscr{E}(\alpha, \mu)$, there is a Δ-system $\mathscr{E}(\alpha, \mu, F) \subseteq \mathscr{E}$ with cardinality μ and kernel F such that each member of $\mathscr{E}(\alpha, \mu, F)$ is disjoint from A_α. Put

$$B_{n+1} = \bigcup_{\omega \leqslant \lambda \leqslant \kappa} \bigcup_{x \in B_n} [x]_{R(\lambda^+)}$$

$$\cup\, U\{U\mathscr{E}(\alpha, \mu, F): \alpha < \kappa,\ |A_\alpha| \leqslant \mu \leqslant \kappa,\ F \in \mathscr{E}(\alpha, \mu)\}.$$

Note that, since the members of $\mathscr{E}$ are finite and $|[B_n \backslash A_\alpha]^{<\omega}| \leqslant \kappa (\alpha < \kappa)$, it follows from Lemma 2.1 that $|B_{n+1}| \leqslant \kappa$. The set $B = U_{n<\omega} B_n$ has the desired properties. ∎

3. Matroids

Recall that a *matroid* (or *cycle system of finite type*) on a set V is a family $\mathscr{E} \subseteq [V]^{<\omega}$ such that $V = U\mathscr{E}$, $\mathscr{E} = \mathscr{E}_{\min}$ and the following exchange axiom is satisfied:

(*) *for* $A, B \in \mathscr{E}$, $x \in A \backslash B$ *and* $y \in A \cap B$, *there is* $C \in \mathscr{E}$ *such that* $x \in C \subseteq A \cup B \backslash \{y\}$.

A matroid $\mathscr{E}$ is said to be *binary* if $\mathscr{E}^* = \{U\mathscr{M} : \mathscr{M}$ is a finite matching of $\mathscr{E}\}$ is closed under symmetric differences. For example, the circuits in a graph form a binary matroid.

Clearly, for any set system $\mathscr{E}$, f.m.e. $\Rightarrow$ f.m. In fact, if $\mathscr{E}$ is f.m.e. then $\mathscr{E}$ is *countably matchable*, i.e., for any countable set $A \subseteq V$, there is a matching $\mathscr{M} \subseteq \mathscr{E}$ such that $A \subseteq \bigcup \mathscr{M}$.

Lemma 3.1. *For binary matroids*, f.m.e. $\Leftrightarrow$ f.m.

Proof. Suppose the binary matroid $\mathscr{E}$ is f.m. Let $\mathscr{M}$ be a finite matching in $\mathscr{E}$ and let $x \in V \backslash U\mathscr{M}$. By assumption, there is a finite matching $\mathscr{M}' \subseteq \mathscr{E}$ such that $\{x\} \cup U\mathscr{M} \subseteq U\mathscr{M}'$. Since $\mathscr{E}$ is binary, $U\mathscr{M}' + U\mathscr{M} = U\mathscr{M}' \backslash U\mathscr{M} \in \mathscr{E}^*$. Hence there is $E \in \mathscr{E}$ such that $x \in E$ and $E \cap U\mathscr{M} = \varnothing$. ∎

The following example shows that, for matroids, finite matchability does not even imply countable matchability (in fact, $<\kappa$-matchability does not imply κ-matchability).

Example 3.2. Let $\{A_n : n < \omega\}$ be a family of pairwise disjoint sets each of cardinality three, and let $V = \{a\} \cup U_{n<\omega} A_n$, where $a \notin U_{n<\omega} A_n$. Consider the matroid $\mathscr{E} = U_{n<\omega}[\{a\} \cup A_n]^3 \cup U_{m<n<\omega}\{x \cup y : x \in [A_m]^2, y \in [A_n]^2\}$. It is easily seen that $\mathscr{E}$ is f.m. But $\mathscr{E}$ has no perfect matching, for if $\mathscr{M} \subseteq \mathscr{E}$ is a matching and $a \in E \in \mathscr{M}$, then $E \subseteq \{a\} \cup A_n$ for some $n < \omega$ and the singleton $A_n \backslash E$ cannot be included in $U\mathscr{M}$. ∎

Lemma 3.3. *Let* κ *be an infinite cardinal and let* $\mathscr{E}$ *be a matroid. Then* $\mathscr{E} \cup \mathscr{E}^{(\kappa)}$ *satisfies the exchange axiom* (*).

Proof. Let $A, B \in \mathscr{E} \cup \mathscr{E}^{(\kappa)}$, $x \in A \backslash B$, $y \in A \cap B$. We have to show that there is $C \in \mathscr{E} \cup \mathscr{E}^{(\kappa)}$ such that $x \in C \subseteq A \cup B \backslash \{y\}$.

We may assume that there are Δ-systems $\mathscr{E}_A, \mathscr{E}_B \subseteq \mathscr{E}$ having, respectively, kernels A, B and such that $|\mathscr{E}_A|, |\mathscr{E}_B|$ are either 1 of κ. Without loss of generality we may assume that $E_1 \cap B = E_2 \cap A = A \cap B$ for all $E_1 \in \mathscr{E}_A$ and $E_2 \in \mathscr{E}_B$. For $E_1 \in \mathscr{E}_A$ and $E_2 \in \mathscr{E}_B$, there is $F(E_1, E_2) \in \mathscr{E}$ such that $x \in F(E_1, E_2) \subseteq E_1 \cup E_2 \backslash \{y\}$. If $F(E_1, E_2) \subseteq A \cup B$ for some $E_1 \in \mathscr{E}_A$ and $E_2 \in \mathscr{E}_B$, then we are done. If not, then $|\mathscr{E}_A \cup \mathscr{E}_B| = \kappa$ and there is

$C \subseteq A \cup B \backslash \{y\}$ such that $x \in C$ and C is the kernel of a Δ-system of size κ in $\mathscr{E}$. ▮

COROLLARY 3.4. *If $\mathscr{E}$ is a matroid and $x \in E \in \mathscr{E} \cup \mathscr{E}^{(\kappa)}$, then there is $F \in \mathscr{E}^{\langle\kappa\rangle}$ such that $x \in F \subseteq E$.*

Proof. Let F be a minimal member of $\mathscr{E} \cup \mathscr{E}^{(\kappa)}$ with the property that $x \in F \subseteq E$. Suppose $F \notin \mathscr{E}^{\langle\kappa\rangle}$. Then there is $B \in \mathscr{E}^{\langle\kappa\rangle}$ such that $B \subseteq F$. By the minimality of F, it follows that $x \notin B$. It follows from the lemma that, if $b \in B$, then there is $F_1 \in \mathscr{E} \cup \mathscr{E}^{(\kappa)}$ such that $x \in F_1 \subseteq F \backslash \{b\}$. But this contradicts the minimality of F. Hence $F \in \mathscr{E}^{\langle\kappa\rangle}$. ▮

COROLLARY 3.5. *If $\mathscr{E}$ is a matroid on V, then so also is $\mathscr{E}^{\langle\kappa\rangle}$ for any infinite cardinal κ.*

COROLLARY 3.6. *A matroid has property $Q(\kappa)$ for every $\kappa \geqslant \omega$.*

4. THE MAIN RESULT

We need the following stronger condition than Q. We will say that the set system $\mathscr{E}$ has property $\mathscr{H}$ if: *for every subset $A \subseteq V$, the system $\mathscr{E} \upharpoonright A$ has property $Q(\kappa^+)$ for every infinite cardinal κ such that $\kappa^+ \leqslant |A|$*. Since the property of being a matroid is hereditary (i.e., if $A \subseteq V$, then $\mathscr{E} \upharpoonright A$ satisfies (*) if $\mathscr{E}$ does), it follows from Corollary 3.6 that a matroid has property $\mathscr{H}$. Theorem 1 is thus a special case of

THEOREM 4.1. *If a family of finite sets, $\mathscr{E}$, has property $\mathscr{H}$ and is* f.m.e., *then $\mathscr{E}$ has a perfect matching.*

Note that the condition f.m.e. is only sufficient for matchability, it is not necessary.

EXAMPLE 4.2. Let $V = A \cup \omega$, where $|A| = 4$ and $A \cap \omega = \varnothing$, and let $\mathscr{E} = [A]^3 \cup [\omega]^2 \cup \{X \in [V]^3 \colon |X \cap A| = 2\}$. It is easy to check that $\mathscr{E}$ is a matroid (and so $\mathscr{E} \in \mathscr{H}$) which has a perfect matching, but it is not f.m.e. since $\{B\}$ is not extendable if $B \in [A]^3$.

Proof of Theorem 4.1. The proof is by induction on the cardinality of $V = U\mathscr{E}$. If $|V| \leqslant \omega$, the result is trivial since (as we already observed in the remark preceeding Lemma 3.1) if $\mathscr{E}$ is f.m.e. then it is countably matchable (in this case, the condition $\mathscr{E} \in \mathscr{H}$ is satisfied vacuously). Now assume $|V| > \omega$. There are three cases.

Case 1. $|V| = \kappa^+$ *is a successor cardinal.*

By Lemma 2.3 we can construct sets $B_\alpha (\alpha < \kappa^+)$ such that

(i) $B_\beta \subseteq B_\alpha (\beta < \alpha < \kappa^+)$, $|B_\alpha| = \kappa$ and $V = U\{B_\alpha : \alpha < \kappa^+\}$,

(ii) B_α is closed under $R(\kappa^+)$ $(\alpha < \kappa^+)$,

(iii) $A_\alpha = B_\alpha \backslash \hat{B}_\alpha \in P(\mathscr{E}, \kappa)$, where $\hat{B}_\alpha = U\{B_\beta : \beta < \alpha\}$ $(\alpha < \kappa^+)$.

Since each B_β is closed under $R(\kappa^+)$ for $\beta \leqslant \alpha$ $(< \kappa^+)$, it follows that $\hat{B}_\alpha$ and $A_\alpha = B_\alpha \backslash \hat{B}_\alpha$ are also closed under $R(\kappa^+)$. Therefore, by Lemma 2.2, $\mathscr{E} \restriction A_\alpha$ is f.m.e. and $A_\alpha = U(\mathscr{E} \restriction A_\alpha)$. Also, since the property $\mathscr{H}$ is hereditary, it follows from the induction hypothesis that there is a matching $\mathscr{M}_\alpha$ of $\mathscr{E}$ such that $A_\alpha = U\mathscr{M}_\alpha$. Then $\mathscr{M} = U\{\mathscr{M}_\alpha : \alpha < \kappa^+\}$ is a perfect matching of $\mathscr{E}$.

Case 2. *$|V| = \kappa$ is an uncountable regular limit cardinal.*

In this case we use Lemma 2.3 to construct an increasing closed sequence of sets B_α $(\alpha < \kappa)$ such that (i) $B_0 = \varnothing$ and $|B_\alpha| = \omega_\alpha$ $(1 \leqslant \alpha < \kappa)$, $V = U\{B_\alpha : \alpha < \kappa\}$, (ii) $B_{\alpha+1}$ is closed under $R(\omega_{\beta+1})$ for $\beta \leqslant \alpha + 1 < \kappa$ and (iii) $B_{\alpha+1} \backslash B_\beta \in P(\mathscr{E}, \omega_\Upsilon)$ for $\beta \leqslant \Upsilon \leqslant \alpha + 1 < \kappa$.

Let $S = \{\alpha < \kappa : B_\alpha$ *is closed under* $R(\omega_{\beta+1})$ *for all* $\beta \leqslant \alpha\}$. Thus every successor ordinal belongs to S by (ii). If $\alpha \in \kappa \backslash S$, then, by definition of S, there are $\beta(\alpha) \leqslant \alpha$ and $F_\alpha \in \mathscr{E}^{\langle \omega_{\beta(\alpha)+1} \rangle}$ such that $F_\alpha \cap B_\alpha \neq \varnothing$ and $F_\alpha \backslash B_\alpha \neq \varnothing$. Also, since α is a limit ordinal and $B_\alpha = U\{B_\Upsilon : \Upsilon < \alpha\}$, there is $\Upsilon(\alpha) < \alpha$ such that $F_\alpha \cap B_\alpha = F_\alpha \cap B_{\Upsilon(\alpha)}$. Suppose $\kappa \backslash S$ is stationary in κ. Then there is $S' \in [\kappa \backslash S]^\kappa$ such that $\Upsilon(\alpha) = \Upsilon$ for all $\alpha \in S'$. Also, since $|[B_\Upsilon]^{<\omega}| = \omega_\Upsilon$, we may assume that $F_\alpha \cap B_\Upsilon = F$ for every $\alpha \in S'$. Moreover, since the F_α are finite, we may assume further that the sets $F_\alpha \backslash B_\alpha$ $(\alpha \in S')$ are pairwise disjoint, i.e., the sets F_α $(\alpha \in S')$ form a Δ-system of size κ with kernel F. Choose $\alpha_0 \in S'$ so that $\beta = \beta(\alpha_0) = \min\{\beta(\alpha) : \alpha \in S'\}$. Since, for $\alpha \in S'$, F_α is either a member of $\mathscr{E}_{\min}$ or is the kernel of a Δ-system in $\mathscr{E}$ of size $\omega_{\beta+1}$, it follows that F is also the kernel of a Δ-system in $\mathscr{E}$ of size $\omega_{\beta(\alpha)+1}$. But this contradicts the fact that F_{α_0} is a minimal member of $\mathscr{E} \cup \mathscr{E}^{(\omega_{\beta+1})}$. It follows that $\kappa \backslash S$ is non-stationary in κ.

Since $\kappa \backslash S$ is non-stationary, there is a closed cofinal subset C of κ such that $C \subseteq S$. Let $C = \{\Upsilon_\nu : \nu < \kappa\}$, where $0 = \Upsilon_0 < \Upsilon_1 < \cdots$. Put $A_\nu = B_{\Upsilon_{\nu+1}} \backslash B_{\Upsilon_\nu}$ $(\nu < \kappa)$. Then $V = U\{A_\nu : \nu < \kappa\}$ since C is closed and the sequence B_α $(\alpha < \kappa)$ is closed, and $V = U\{B_\alpha : \alpha < \kappa\}$. Since $A_\nu = U\{B_{\alpha+1} \backslash B_{\Upsilon_\nu} : \Upsilon_\nu \leqslant \alpha < \Upsilon_{\nu+1}\}$ $(\nu < \kappa)$, and since $B_{\alpha+1} \backslash B_{\Upsilon_\nu} \in P(\mathscr{E}, \omega_{\Upsilon_\nu})$ by (iii), it follows that $A_\nu \in P(\mathscr{E}, \omega_{\Upsilon_\nu})$ $(\nu < \kappa)$. Also, since Υ_ν and $\Upsilon_{\nu+1}$ belong to S, both B_{Υ_ν} and $B_{\Upsilon_{\nu+1}}$ are closed under $R(\omega_{\Upsilon_\nu+1})$, and therefore so also is A_ν. It now follows by Lemma 2.2 that $A_\nu = U(\mathscr{E} \restriction A_\nu)$ and $\mathscr{E} \restriction A_\nu$ is f.m.e. Therefore, by the induction hypothesis (since $\mathscr{E} \restriction A_\nu \in \mathscr{H}$), there is a perfect matching $\mathscr{M}_\nu$ of $\mathscr{E} \restriction A_\nu$ $(\nu < \kappa)$ and then $\mathscr{M} = U\{\mathscr{M}_\nu : \nu < \kappa\}$ is a perfect matching of $\mathscr{E}$.

Case 3. $|V| = \kappa$ *is a singular cardinal.*

The proof for this part of theorem resembles the proof for the singular case of the "marriage theorem" of Aharoni *et al.* [1], this type of proof was first used by Shelah in [6].

Let κ_ν $(\nu < cf(\kappa))$ be a closed strictly increasing sequence of cardinals cofinal in κ. We assume $\kappa_0 > cf(\kappa)$. We may write $V = U\{A_\nu^0 : \nu < cf(\kappa)\}$, where $A_0^0 \subseteq A_1^0 \subseteq \cdots \subseteq A_\nu^0 \subseteq \cdots$ and $|A_\nu^0| = \kappa_\nu$. We shall define by transfinite induction subsets $A_\nu^n \subseteq V$ for $n < \omega$ and $\nu < cf(\kappa)$ such that $|A_\nu^n| = \kappa_\nu$, $A_\nu^0 \subseteq A_\nu^1 \subseteq \cdots \subseteq A_\nu^n \subseteq \cdots$ and $A_0^n \subseteq A_1^n \subseteq \cdots \subseteq A_\nu^n \subseteq \cdots$.

Let $n > 0$ and $\nu_0 < cf(\kappa)$, and suppose that we have already defined the sets A_ν^{n-1} for $\nu < cf(\kappa)$ and A_ν^n for $\nu < \nu_0$. We have to define $A_{\nu_0}^n$. First, for each $\nu < cf(\kappa)$, let $\{x_{\nu,\rho}^{n-1} : \rho < \kappa_\nu\}$ be any well-ordering of the elements of A_ν^{n-1} in type κ_ν. Put

$$B_{\nu_0}^n = A_{\nu_0}^{n-1} \cup U\{A_\nu^n : \nu < \nu_0\} \cup \{x_{\nu,\rho}^{n-1} : \nu_0 < \nu < cf(\kappa) \text{ and } \rho < \kappa_{\nu_0}\}.$$

Now define a set $C_{\nu_0}^n$ as follows. If ν_0 is a limit ordinal, we simply put $C_{\nu_0}^n = B_{\nu_0}^n$. Suppose $\nu_0 = \nu_1 + 1$ is a successor ordinal. For every $F \in \mathscr{E}^{(\kappa_{\nu_1}^+)} \cap [B_{\nu_0}^n \backslash A_{\nu_1}^n]^{<\omega}$ choose a Δ-system, $\mathscr{E}(F) \subseteq \mathscr{E}$, with kernel F and having cardinality $|\mathscr{E}(F)| = \kappa_{\nu_1}^+$ and put

$$C_{\nu_0}^n = B_{\nu_0}^n \cup U\{U\mathscr{E}(F) : F \in \mathscr{E}^{(\kappa_{\nu_1}^+)} \cap [B_{\nu_0}^n \backslash A_{\nu_1}^n]^{<\omega}\}.$$

Finally, we set

$$A_{\nu_0}^n = \bigcup_{\nu \leqslant \nu_0} \{U[x]_{F(\kappa_\nu^+)} : x \in C_{\nu_0}^n\}.$$

Thus $A_{\nu_0}^{n-1} \subseteq B_{\nu_0}^n \subseteq C_{\nu_0}^n \subseteq A_{\nu_0}^n$ and $|A_{\nu_0}^n| = \kappa_{\nu_0}$ by Lemma 2.1. This defines the A_ν^n for $n < \omega$ and $\nu < cf(\kappa)$, and we put

$$A_\nu = U\{A_\nu^n : n < \omega\}.$$

It is clear from the definitions that $|A_\nu| = \kappa_\nu$ $(\nu < cf(\kappa))$ and $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\nu \subseteq \cdots$. We claim that the increasing sequence of sets $(A_\nu : \nu < cf(\kappa))$ is actually closed. To see this, suppose that ν_0 is a limit ordinal and that $x \in A_{\nu_0}$. Then $x \in A_{\nu_0}^n$ for some $n < \omega$ and so $x = x_{\nu_0,\rho}^n$ for some $\rho < \kappa_{\nu_0}$. Since $\kappa_{\nu_0} = \sum\{\kappa_\nu : \nu < \nu_0\}$, there is some $\nu_1 < \nu_0$ such that $\rho < \kappa_{\nu_1}$ and hence $x \in B_{\nu_1}^{n+1} \subseteq A_{\nu_1}^{n+1} \subseteq A_{\nu_1}$. Thus $A_{\nu_0} = U\{A_\nu : \nu < \nu_0\}$. It follows that $V = U\{V_\nu : \nu < cf(\kappa)\}$, where $V_\nu = A_{\nu+1} \backslash A_\nu$.

By construction, the set A_ν^n is closed under $R(\kappa_\rho^+)$ for $\rho \leqslant \nu < cf(\kappa)$ and $n < \omega$. Thus A_ν is also closed under $R(\kappa_\rho^+)$ for $\rho \leqslant \nu < cf(\kappa)$. It follows that V_ν is closed under $R(\kappa_\nu^+)$ $(\nu < cf(\kappa)$. We claim also that $V_\nu \in P(\mathscr{E}, \kappa_\nu)$. Indeed, if $F \in \mathscr{E}^{(\kappa_\nu^+)} \cap [V_\nu]^{<\omega}$, then $F \subseteq B_{\nu+1}^n \backslash A_\nu$ for some $n < \omega$. Now, by the definition of $C_{\nu+1}^n$, F is the kernel of a Δ-system $\mathscr{E}(F) \subseteq \mathscr{E} \restriction C_{\nu+1}^n$ of size

$|\mathscr{E}(F)| = \kappa_\nu^+$. Since $|A_\nu| = \kappa_\nu$ and $F \cap A_\nu = \emptyset$, it follows that F is also the kernel of a Δ-system in $\mathscr{E} \upharpoonright V_\nu$ of size κ_ν^+.

Since $V_\nu \in P(\mathscr{E}, \kappa_\nu)$ and V_ν is closed under $R(\kappa_\nu^+)$ $(\nu < cf(\kappa))$, it follows from Lemma 2.2 that $V_\nu = U(\mathscr{E} \upharpoonright V_\nu)$ and $\mathscr{E} \upharpoonright V_\nu$ is f.m.e. By the hereditary nature of $\mathscr{H}$, we have $\mathscr{E} \upharpoonright V_\nu \in \mathscr{H}$ and so, by the induction hypothesis, $\mathscr{E} \upharpoonright V_\nu$ has a perfect matching $\mathscr{M}_\nu$ $(\nu < cf(\kappa))$ and $\mathscr{M} = U\{\mathscr{M}_\nu : \nu < cf(\kappa)\}$ is a perfect matching of $\mathscr{E}$. ∎

5. A Further Result for Matroids

The example, Example 3.2, that we gave of an f.m. matroid that is not matchable is, in a certain sense, the simplest possible such example. In this section we shall prove

THEOREM 5.1. *Let $\mathscr{E}$ be a matroid on V such that $\mathscr{E} \setminus S \subseteq [V \setminus S]^{\leqslant 3}$ for some finite set $S \subseteq V$. Then $\mathscr{E}$ is matchable if and only if it is finitely matchable.*

We need some preliminary lemmas.

A matroid $\mathscr{E}$ is *connected* if and only if the hypergraph $(V, \mathscr{E})$ is connected, i.e., for any $a, b \in V$ there is a finite sequence $E_0, E_1, ..., E_n$ of members of $\mathscr{E}$ such that $a \in E_0$, $b \in E_n$ and $E_i \cap E_{i+1} \neq \emptyset$ $(i < n)$. The following result is known and easily proved (see, e.g., [2, 6.83; 7, 5.2]).

LEMMA 5.2. *For any pair of vertices a, b of a connected matroid $\mathscr{E}$, there is $E \in \mathscr{E}$ such that $\{a, b\} \subseteq E$.*

LEMMA 5.3. *If S is a finite set of vertices of a connected matroid $\mathscr{E}$, then $\mathscr{E} \setminus S$ has only finitely many connected components.*

Proof. Suppose that $\mathscr{E}_i$ $(i \in I)$ are the connected components of $\mathscr{E} \setminus S$, and let $V_i = U\mathscr{E}_i$. For $E \in \mathscr{E}$, let $I(E) = \{i \in I : E \cap V_i \neq \emptyset\}$, and let $\mathscr{E}' = \{E \in \mathscr{E} : |I(E)| \geqslant 2\}$. Then $E \cap S \neq \emptyset$ for all $E \in \mathscr{E}'$. Suppose for contradiction that I is infinite. By Lemma 5.2, it follows that there is an infinite Δ-system $\mathscr{E}_1 \subseteq \mathscr{E}'$ with kernel $T \subseteq S$ such that $I(E_1) \cap I(E_2) = \emptyset$ whenever E_1, E_2 are distinct members of $\mathscr{E}_1$. We may assume that T is a minimal kernel of an infinite Δ-system of this kind. Clearly $T \neq \emptyset$. Let $t \in T$. For a pair of distinct members E_1, E_2 of $\mathscr{E}_1$, there is $E \in \mathscr{E}$ such that $E \subseteq E_1 \cup E_2 \setminus \{t\}$. In fact, $E \in \mathscr{E}'$. It follows that there is an infinite Δ-system $\mathscr{E}_2 \subseteq \mathscr{E}'$ with kernel $T' \subseteq T \setminus \{t\}$ which also has the property that $I(E) \cap I(E') = \emptyset$ for distinct members E, E' of $\mathscr{E}_2$. This contradicts the assumed minimality of T and hence I is finite. ∎

LEMMA 5.4. *If $\mathscr{E}$ is a matroid and $[V\backslash S]^1 \subseteq \mathscr{E}^{\langle\omega\rangle}$ for some finite set $S \subseteq V$, then $\mathscr{E}$ is matchable if and only if $\mathscr{E}$ is* f.m.

Proof. Suppose $\mathscr{E}$ is finitely matchable. Then there is a finite matching $\mathscr{M} \subseteq \mathscr{E}$ such that $S \subseteq U\mathscr{M}$. The hypothesis implies that $\mathscr{E}' = \mathscr{E}\backslash U\mathscr{M}$ is f.m.e. and $U\mathscr{E}' = V\backslash U\mathscr{M}$. By Theorem 1, there is a perfect matching $\mathscr{M}'$ of $\mathscr{E}'$ and $\mathscr{M} \cup \mathscr{M}'$ is a perfect matching of $\mathscr{E}$. ∎

The next two lemmas are special for matroids $\mathscr{E}$ satisfying $\mathscr{E} \subseteq [V]^{\leqslant 3}$.

LEMMA 5.5. *If $\mathscr{E}$ is an infinite connected matroid and $\mathscr{E} \subseteq [V]^{\leqslant 3}$, then $\mathscr{E}^{\langle\omega\rangle} \subseteq [V]^{\leqslant 2}$.*

Proof. Suppose for contradiction that $E = \{x, y, z\}$ is a memeber of $\mathscr{E}^{\langle\omega\rangle}$ of size $|E| = 3$. By Lemma 5.2 and since $\{x\} \notin \mathscr{E}^{\langle\omega\rangle}$, there is some $a \in V\backslash E$ such that $\{a, x\} \in \mathscr{E}^{(\omega)}_{\min}$. By Corollary 3.5, there is $E' \in \mathscr{E}^{\langle\omega\rangle}$ such that $a \in E' \subseteq E \cup \{a\}\backslash\{x\}$. If $E' = \{a, y\}$, then again by 3.5, there is $E'' \subseteq \{x, y\}$ such that $E'' \in \mathscr{E}^{\langle\omega\rangle}$ and this contradicts the fact that E is a minimal member of $\mathscr{E}^{\langle\omega\rangle}$. Thus $E' \neq \{a, y\}$, and similarly $E' \neq \{a, z\}$. Therefore, $E' = \{a, y, z\} \in \mathscr{E}$. Hence there is $E_1 \in \mathscr{E}$ such that $a \in E_1 \subseteq E \cup E'\backslash\{y\}$. Since $\{a, x\} \notin \mathscr{E}$ and $\{a, z\} \subseteq E'$, it follows that $E_1 = \{a, x, z\}$. Since $E_1 \notin \mathscr{E}^{\langle\omega\rangle}$ and $\mathscr{E}^{\langle\omega\rangle} \in Q(\omega)$ (by 3.5 and 3.6), it follows that either $\{z\}$ or $\{x, z\} \in \mathscr{E}^{\langle\omega\rangle}$ (if $\{a, z\} \in \mathscr{E}^{\langle\omega\rangle}$ this implies that $\{z\}$ or $\{x, z\} \in \mathscr{E}^{\langle\omega\rangle}$). However, this contradicts the fact that E is a minimal member of $\mathscr{E}^{\langle\omega\rangle}$. Therefore $E \notin \mathscr{E}^{\langle\omega\rangle}$. ∎

LEMMA 5.6. *If $\mathscr{E}$ is an infinite connected matroid and $\mathscr{E} \subseteq [V]^{\leqslant 3}$, then there is a finite set $S \subseteq V$ such that $\mathscr{E}^{\langle\omega\rangle} = [S]^2 \cup [V\backslash S]^1$.*

Proof. By Lemma 5.5, the matroid $\mathscr{E}^{\langle\omega\rangle} \subseteq [V]^{\leqslant 2}$ and, therefore, the connected components of $\mathscr{E}^{\langle\omega\rangle}$ are either singletons or complete graphs.

Let $S = \{x \in V\colon \{x\} \notin \mathscr{E}^{(\omega)}\}$. By Lemma 5.2, if $x, y \in S$, there is $E \in \mathscr{E}$ such that $\{x, y\} \subseteq E$. If $E = \{x, y\}$, then $E \in \mathscr{E}^{\langle\omega\rangle}$. Suppose $E = \{x, y, z\}$. Since $\mathscr{E} \in Q(\omega)$, it follows that either (i) $\{x, y\} \in \mathscr{E}^{\langle\omega\rangle}$ or (ii) both $\{x, z\}$ and $\{y, z\} \in \mathscr{E}^{\langle\omega\rangle}$. But (ii) implies (i). Thus, in any case, $\{x, y\} \in \mathscr{E}^{\langle\omega\rangle}$ and so S is the only non-trivial component of $\mathscr{E}^{\langle\omega\rangle}$. Moreover, since $\{x\} \notin \mathscr{E}^{(\omega)}$ for $x \in S$, it follows that S must be finite. ∎

Proof of Theorem 5.1. We may assume that $\mathscr{E}$ is connected and infinite and finitely matchable. By Lemma 5.3, $\mathscr{E}\backslash S$ has only finitely many components. Let $\mathscr{E}_i$ $(i \in I_1)$ be the finite components and $\mathscr{E}_i$ $(i \in I_2)$ the infinite components of $\mathscr{E}\backslash S$. By Lemma 5.6, for each $i \in I_2$, there is a finite set $S_i \subseteq U\mathscr{E}_i = V_i$ such that $[V_i\backslash S_i]^1 \subseteq \mathscr{E}^{\langle\omega\rangle}$. Therefore, $[V\backslash S']^1 \subseteq \mathscr{E}^{\langle\omega\rangle}$, where $S' = S \cup U\{U\mathscr{E}_i\colon\ i \in I_1\} \cup U\{S_i\colon i \in I_2\}$ is finite, and hence $\mathscr{E}$ is matchable by Lemma 5.4. ∎

REFERENCES

1. R. AHARONI, C. ST. J. A. NASH-WILLIAMS, AND S. SHELAH, A general criterion for the existence of transversals, *Proc. London Math. Soc. (3)* **47** (1983), 43–68.
2. M. AIGNER, "Combinatorial Theory," Springer-Verlag, Berlin, 1979.
3. C. ST. J. A. NASH-WILLIAMS, Decomposition of graphs into closed and endless chains, *Proc. London Math. Soc.* **10** (1960), 221–238.
4. N. POLAT, A compactness result for perfect matchings, *Discrete Math.* **64** (1987), 229–237.
5. G. SABIDUSSI, "Compactness in Graphs," Lecture at the Canad. Math. Congress meeting, Waterloo, 1971.
6. S. SHELAH, A compactness theorem for singular cardinals, free algebra, Whitehead problem and transversals, *Israel J. Math.* **21** (1975), 319–349.
7. D. J. A. WELSH, "Matroid Theory," Academic Press, London, 1976.